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Abstract: We consider a financial market model with a single risky asset whose price process evolves according to a general jump-diffusion with locally bounded coefficients and where market participants have only access to a partial information flow $(\mathcal{E}_t)_{t\geq 0} \subseteq (\mathcal{F}_t)_{t\geq 0}$. For any utility function, we prove that the partial information financial market is locally viable, in the sense that the problem of maximizing the expected utility of terminal wealth has a solution up to a stopping time, if and only if the marginal utility of the terminal wealth is the density of a partial information equivalent martingale measure (PIEMM). This equivalence result is proved in a constructive way by relying on maximum principles for stochastic control under partial information. We then show that the financial market is globally viable if and only if there exists a partial information local martingale deflator (PILMD), which can be explicitly constructed. In the case of bounded coefficients, the latter turns out to be the density process of a global PIEMM. We illustrate our results by means of an explicit example.

Key-words: Optimal portfolio, jump-diffusion, partial information, maximum principle, BSDE, viability, martingale measure, utility maximization

[0] **MSC (2010):** 60G44, 60G51, 60G57, 91B70, 91G80, 93E20, 94A17

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Viabilté et mesures martingales en information partielle

Résumé : On considère un marché financier constitué d'un actif risqué dont le prix est modélisé par un processus de diffusion avec saut à coefficients bornés. On suppose que les investisseurs n'ont accès qu'à une information partielle $(\mathcal{E}_t)_{t\geq 0} \subseteq (\mathcal{F}_t)_{t\geq 0}$. Pour toute fonction d'utilité on démontre que ce marché est localement viable, dans le sens où le problème de maximisation de l'espérance de l'utilité de la richesse terminale a une solution jusqu'à un temps d'arrêt si et seulement si l'utilité marginale de la richesse terminale est la densité d'une mesure martingale équivalente sous information partielle. On prouve cette équivalence de manière constructive au moyen de principes de maximum stochastique pour le contrôle en information partielle. On démontre ensuite que le marché financier est globalement viable si et seulement s'il existe un déflateur martingale local sous information partielle, que l'on peut construire explicitement. Dans le cas de coefficients bornés, ce dernier correspond au processus de densité d'une mesure martingale équivalente sous information partielle. Nous illustrons ces résultats sur un exemple.

Mots-clés : Optimisation de portefeuilles, diffusion avec sauts, information partielle, principe du maximum, EDSR, viabilité, mesure martingale, maximisation d'utilité.

1 Introduction

The concepts of no-arbitrage, martingale measure and portfolio optimization can be rightly considered as the cornerstones of modern mathematical finance, starting from the seminal papers [12] and [18]. Loosely speaking, the no-arbitrage paradigm is equivalent to the existence of a martingale measure, which can then be used for pricing purposes (risk-neutral valuation), and, again loosely speaking, portfolio optimization problems are solvable if and only if the financial market does not admit arbitrage opportunities.

In the context of discrete-time models on a finite probability space (see e.g. [22] and [23]), it can actually be shown that the above concepts are equivalent and, furthermore, one can work out explicitly the connections between them. Indeed, it is well-known that there exists an Equivalent Martingale Measure (EMM) if and only if there do not exist arbitrage opportunities, which in turn is equivalent to the solvability of portfolio optimization problems. In particular, one can obtain an EMM by taking the (normalised) marginal utility of the optimal terminal wealth of a portfolio optimization problem. This relation also represents a classical and well-known result from the economic literature (see e.g. [3], Section 4.4). In the case of discrete-time models on a general probability space, the validity of this equivalence has been studied in [26], [30] and [31].

When one moves from discrete-time to continuous-time financial models, then things become quickly more complicated and the equivalences discussed so far do not hold any more in full generality. For instance, in order to recover the equivalence between EMMs and no-arbitrage, one has to replace the notion of martingale with the notion of *local* martingale (or even σ-martingale, see [7]) and the condition of no-arbitrage with the *No Free Lunch with Vanishing Risk (NFLVR)* condition (see [5]-[7]). Furthermore, the marginal utility of the optimal terminal wealth of a portfolio optimization problem does not necessarily yield the density of an equivalent (local-/σ-)martingale measure but only a supermartingale deflator (see e.g. [17] and [29]).

In recent years, there has been a considerable interest in financial market models which go beyond the traditional setting based on equivalent martingale measures, by relaxing the NFLVR requirement. One of the first studies in this direction is the paper [19], where the authors are concerned with the viability of a complete Itô-process model of a financial market. In particular, they show that the financial market can be viable, in the sense that portfolio optimization problems can be meaningfully solved, even if the NFLVR condition does not necessarily hold. In an analogous context, [11] prove that the notions of completeness and viability do not rely on the existence of equivalent (local) martingale measures. In a general semimartingale framework, Proposition 4.19 of [15] shows that the minimal no-arbitrage requirement in order to solve expected utility maximization problems amounts to the *No Unbounded Profit with Bounded Risk (NUPBR)* condition, the latter being weaker than NFLVR. However, in all these works, there is no general and explicit connection between the solvability of a portfolio optimization problem and the density of a candidate martingale measure.

In the present paper, we consider a rather general jump-diffusion model, with locally bounded coefficients, and study the issue of the *viability* of the financial market, defined as the ability to solve a portfolio optimization problem. Our main goal consists in characterising the notion of viability in terms of martingale measures, in a suitable sense to be made precise in the following. A distinguishing feature of our approach is that we refrain from imposing a-priori any no-arbitrage restriction on the model, tackling instead directly the solvability of portfolio optimization problems. Furthermore, we suppose that market participants have only access to a partial information flow, which does not contain the whole information of the stochastic basis. In order to solve portfolio optimization problems under partial information, we shall employ necessary and sufficient maximum principles for stochastic control under partial information, as discussed in [2] (see also the recent paper [21] for related results in the complete information case). This approach allows to characterise the optimal solution via an associated BSDE, which in turn requires a good control on the integrability properties of the processes involved. Since we work with quite general processes, we cannot in general guarantee that these integrability conditions are satisfied and, hence, we need to resort to a localization procedure, as explained in Section 3.

The main contributions of the present paper can be outlined as follows:

(i) we show that the financial market where agents trade under partial information is *locally viable*, in the sense that a portfolio optimisation problem admits a solution up to a stopping time, if and only if there exists a *Partial Information Equivalent Martingale Measure (PIEMM)* up to a stopping time. Furthermore, the density of such PIEMM is given by the (normalised) marginal utility of the optimal terminal wealth, thus recovering the classical result of financial economics;

(ii) we prove that the financial market where agents trade under partial information is *globally viable*, in the sense that it is locally viable for a whole sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of increasing stopping times in a consistent way, if and only if there exists a *Partial Information Local Martingale Deflator (PILMD)*. Furthermore, we show that such PILMD can be explicitly constructed by aggregating the densities of all PIEMMs obtained locally;

(iii) in the more specific case where the jump-diffusion describing the price process has bounded coefficients, we prove that the financial market is viable on the global time horizon if and only if the (normalised) marginal utility of the optimal terminal wealth defines a PIEMM on the global time horizon. The density process of the latter also coincides with the PILMD obtained in (ii);

(iv) by means of an explicit example (see Section 5) we show that, even for a regular utility function and continuous-path processes with good integrability properties but unbounded coefficients, one cannot do better than (ii) in the global case, in the sense that a PIEMM may fail to exist globally.

To the best of our knowledge, the issue of linking the viability of the financial market to the existence of (weaker counterparts of) equivalent martingale measures such as PIEMMs and PILMDs has never been dealt with in the partial information case. Furthermore, even in the full information case (i.e., letting $\mathbb{E} = \mathbb{F}$, according to the notation introduced in Section 2) we go significantly beyond a pure existence result, since our approach allows to obtain a precise and explicit connection between the solution of an optimal portfolio problem and a density of an equivalent martingale measure / local martingale deflator, in a local and in a global sense (see Sections 3 and 4, respectively). We refer to Remark 3.10 for a comparison of our work with the duality theory developed in [17] and [29].

The paper is structured as follows. Section 2 contains a general description of the modelling framework. In Section 3, we prove the equivalence between local market viability and the existence of a PIEMM up to a stopping time. More specifically, this requires first to characterise all optimal portfolios in terms of the solution to an associated BSDE (Section 3.1) and then to characterise the family of density processes of all PIEMMs (Section 3.2), the main equivalence result is then proved in Section 3.3. Section 4 deals with the issue of the global viability of the financial market, first in the simpler case of bounded coefficients (Section 4.1) and then in the more general locally bounded case (Section 4.2). Section 5 closes the paper and presents an explicit example with the purpose of illustrating some of the main concepts and results.

2 The modelling framework

On a given probability space $(\Omega, \mathcal{F}, P)$, let us consider a Brownian motion $B = \{B(t); t \geq 0\}$ and a homogeneous Poisson random measure $N(\cdot, \cdot)$ on $\mathbb{R}_+ \times \mathbb{R}$, in the sense of Definition II.1.20 of [14], independent of B. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by B and N, assumed to satisfy the usual conditions of right-continuity and P-completeness, and denote by $\mathcal{P}_{\mathbb{F}}$ the predictable σ-field of $\mathbb{F}$. We denote by $m(dt, d\zeta) := \nu(d\zeta)\, dt$ the compensator of the random measure $N(dt, d\zeta)$, where ν is a σ-finite measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$, and by $\widetilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)\, dt$ the corresponding compensated random measure. Finally, we let $T \in (0, \infty)$ represent a fixed time horizon.

We consider an abstract financial market with two investment possibilities (all the results of the present paper can be generalised to multi-dimensional market models without significant difficulties):

(i) a risk-free asset with unit price $S^0(t) = 1$, for all $t \in [0, T]$;

(ii) a risky asset, with unit price $S(t)$ given by the solution to the stochastic differential equation

$$\begin{cases} dS(t) = b(t)\, dt + \sigma(t)\, dB(t) + \displaystyle\int_{\mathbb{R}} \gamma(t, \zeta)\, \widetilde{N}(dt, d\zeta)\ , \quad t \in [0, T]; \\ S(0) = S_0 > 0. \end{cases} \tag{2.1}$$

We assume that the processes $b = \big\{b(t); t \in [0, T]\big\}$ and $\sigma = \big\{\sigma(t); t \in [0, T]\big\}$ are $\mathcal{P}_{\mathbb{F}}$-measurable and locally bounded and that $\gamma : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a predictable function in the sense of [14], i.e., it is $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable, and, for any $D \in \mathcal{B}(\mathbb{R})$, the process $\gamma(\cdot, D) = \big\{\gamma(t, D); t \in [0, T]\big\}$ is locally bounded. We refer the reader to Chapter II of [14] and to the monograph [20] for more information on stochastic calculus with respect to Poisson random measures.

Remark 2.1. The local boundedness assumption implies that there exists a sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n \in \mathbb{N}}$ with $\tau_n \nearrow +\infty$ P-a.s. as $n \to +\infty$ such that the stopped processes S^{τ_n},

b^{τ_n}, σ^{τ_n} and $\gamma(\cdot\wedge\tau_n,D)$, for $D\in\mathcal{B}(\mathbb{R})$, are P-a.s. uniformly bounded, for all $n\in\mathbb{N}$. Note that this assumption is always satisfied if the processes b, σ and $\gamma(\cdot,D)$, for all $D\in\mathcal{B}(\mathbb{R})$, are left-continuous or right-continuous with limits from the left (see e.g. [13], Theorem 7.7).

As mentioned in the introduction, we shall be interested in a financial market model where market participants do not have access to the *full information* filtration $\mathbb{F}$. To this effect, we introduce a filtration $\mathbb{E}=(\mathcal{E}_t)_{0\leq t\leq T}$, which represents the *partial information* actually available. We assume that $\mathbb{E}$ satisfies the usual conditions and that $\mathcal{E}_t\subseteq\mathcal{F}_t$ for all $t\in[0,T]$. For example, we could have:

(i) $\mathcal{E}_t=\mathcal{F}_{(t-\delta)^+}$, for some $\delta>0$, representing a delayed information flow;

(ii) $\mathcal{E}_t=\bar{\mathcal{F}}^S_t$, where $\bar{\mathcal{F}}^S_t$ is the σ-algebra generated by $\{S(t_i);0=t_0<t_1<\ldots<t_n\leq t\}$, $n\in\mathbb{N}$, representing the information flow generated by discrete observations of the price process S;

(iii) $\mathcal{E}_t=\mathcal{F}^S_t$, where $\mathcal{F}^S_t$ is the σ-algebra generated by $\big\{S(u);u\in[0,t]\big\}$, representing the information flow generated by continuous observations of the price process S.

Note that, in the cases (ii)-(iii), the filtration $\mathbb{E}$ is in general strictly smaller than $\mathbb{F}$, since the observation of the price process S does not permit to unveil the sources of randomness B and N.

We say that a given function $U:(-\infty,\infty]\to[-\infty,\infty)$ of class $\mathcal{C}^1$ on $(-\infty,\infty)$ is a *utility function* if it is concave and strictly increasing on $(-\infty,\infty]$ and we denote by U' its first derivative (*marginal utility*). Aiming at describing the activity of trading in the financial market on the basis of the information represented by the filtration $\mathbb{E}$ and according to the preference structure represented by U, we define the family $\mathcal{A}^U_{\mathbb{E}}$ of *admissible strategies* as follows, for some $\lambda,\mu>2$ with $2/\lambda+2/\mu=1$:

$$\mathcal{A}^U_{\mathbb{E}}:=\big\{\text{all }\mathbb{E}\text{-predictable processes }\varphi=\big\{\varphi(t);t\in[0,T]\big\}\text{ s.t. }X_\varphi\in\mathcal{S}^\lambda\text{ and }E\big[U'\big(X_\varphi(T)\big)^\mu\big]<\infty\big\}$$

where φ_t represents the number of units of the risky asset held in the portfolio at time t, for all $t\in[0,T]$, with associated wealth process $X_\varphi=\big\{X_\varphi(t);t\in[0,T]\big\}$, and where $\mathcal{S}^\lambda$ denotes the family of all semimartingales $Y=\big\{Y(t);t\in[0,T]\big\}$ satisfying $E\big[\sup_{t\in[0,T]}|Y(t)|^\lambda\big]<\infty$. The requirement of $\mathbb{E}$-predictability amounts to ensuring that agents trade by relying only on the information at their disposal. As usual, we assume that trading is done in a self-financing way, so that the wealth process associated to a given strategy $\varphi\in\mathcal{A}^U_{\mathbb{E}}$ starting from an initial endowment $x\in\mathbb{R}$ is given by:

$$\begin{cases} dX_\varphi(t)=\varphi(t)\,dS(t)=\varphi(t)\left(b(t)\,dt+\sigma(t)\,dB(t)+\displaystyle\int_{\mathbb{R}}\gamma(t,\zeta)\,\widetilde{N}(dt,d\zeta)\right), \quad t\in[0,T]; \\ X_\varphi(0)=x. \end{cases} \tag{2.2}$$

We want to emphasise that we did not introduce a-priori any no-arbitrage restriction on the financial market model. In the remaining part of the paper, the no-arbitrage properties of the model will be inferred as the consequence of the (local) solvability of a portfolio optimization problem.

3 Local market viability under partial information

In the present section we prove the equivalence between the concept of local market viability, introduced below in Definition 3.2, and the local existence of a Partial Information Equivalent Martingale Measure (see Definition 3.6) such that its density is expressed in terms of the (normalised) marginal utility of terminal wealth. The reason why we embark on a local analysis is due to the fact that, aiming at a characterisation of the solutions to portfolio optimisation problems, we want to apply the maximum principles for stochastic control under partial information developed in [2]. However, in order to have a good control on the integrability properties of the processes involved, we first need to localize and rely on the local boundedness assumption on b, σ and γ (see Remark 2.1). Under a more restrictive assumption, a direct global result will be proved in Section 4.1.

Problem 3.1 (Partial information locally optimal portfolio problem)**.** For a fixed $n \in \mathbb{N}$, for a given utility function U and an initial endowment $x \in \mathbb{R}$, find an element $\varphi^{*,n} \in \mathcal{A}^U_{\mathbb{E}}(n)$ such that

$$\sup_{\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)} E\Big[U\big(X_\varphi(T \wedge \tau_n)\big)\Big] = E\Big[U\big(X_{\varphi^{*,n}}(T \wedge \tau_n)\big)\Big] < \infty$$

where $\mathcal{A}^U_{\mathbb{E}}(n) := \Big\{\text{all } \mathbb{E}\text{-predictable processes } \varphi = \big\{\varphi(t); t \in [0,T]\big\} \text{ s.t. } \varphi \mathbf{1}_{[\![0,\tau_n]\!]} \in \mathcal{A}^U_{\mathbb{E}}\Big\}$.

Definition 3.2 (Local market viability)**.** *Let U be a utility function and $n \in \mathbb{N}$. The financial market is said to be* locally viable up to τ_n *if Problem 3.1 admits an optimal solution $\varphi^{*,n} \in \mathcal{A}^U_{\mathbb{E}}(n)$.*

Until the end of Section 3, we fix an element $n \in \mathbb{N}$ and let τ_n be the corresponding $\mathbb{F}$-stopping time from the sequence $\{\tau_n\}_{n \in \mathbb{N}}$ introduced in Remark 2.1.

3.1 A BSDE characterisation of locally optimal portfolios

As a first step, we provide a characterisation of the locally optimal portfolio which solves Problem 3.1 in terms of the solution to a Backward Stochastic Differential Equation (BSDE), by relying on the necessary and sufficient maximum principles for stochastic control under partial information established in [2] (see also the recent paper [21] for related results). We denote by b^n the stopped process $b^n := \big\{b(t \wedge \tau_n); t \in [0,T]\big\}$, with an analogous notation for σ^n and γ^n.

We define the Hamiltonian $H^n : \Omega \times [0,T] \times \mathbb{R}^3 \times \mathcal{R} \to \mathbb{R}$ as follows:

$$H^n\big(\omega, t, \varphi, p, q, r(\cdot)\big) := \varphi\, b^n(\omega,t)\, p + \varphi\, \sigma^n(\omega,t)\, q + \varphi \int_{\mathbb{R}} r(\zeta)\, \gamma^n(\omega,t,\zeta)\nu(d\zeta) \tag{3.1}$$

where $\mathcal{R}$ is defined as the class of functions $r : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that the integral in (3.1) converges. To the Hamiltonian H^n we associate a BSDE for the adjoint processes $p^n = \big\{p^n(t); t \in$

$[0,T]\big\}$, $q^n = \big\{q^n(t); t \in [0,T]\big\}$ and for the function $r^n : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ as follows, for any $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$:

$$\begin{cases} dp^n(t) = q^n(t)\, dB(t) + \int_{\mathbb{R}} r^n(t,\zeta)\, \widetilde{N}(dt,d\zeta), \quad t \in [0,T]; \\ p^n(T) = U'\big(X_\varphi(T \wedge \tau_n)\big). \end{cases} \tag{3.2}$$

In order to study the BSDE (3.2), we need to introduce the following classes of processes:

$$\mathcal{M}^\mu := \left\{ \text{all } \mathbb{F}\text{-martingales } M = \big\{M(t); t \in [0,T]\big\} \text{ s.t. } E\Big[\sup_{t\in[0,T]} |M(t)|^\mu\Big] < \infty \right\};$$

$$L^\mu(B) := \left\{ \text{all } \mathbb{F}\text{-predictable processes } q = \big\{q(t); t \in [0,T]\big\} \text{ s.t. } E\left[\left(\int_0^T q^2(t)dt\right)^{\frac{\mu}{2}}\right] < \infty \right\};$$

$$G^\mu(\widetilde{N}) := \left\{ \text{all } \mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})\text{-measurable functions } r \text{ s.t. } E\left[\left(\int_0^T \int_{\mathbb{R}} r^2(t,\zeta)\nu(d\zeta)dt\right)^{\frac{\mu}{2}}\right] < \infty \right\}.$$

Lemma 3.3. *For any fixed $n \in \mathbb{N}$ and $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$, the BSDE* (3.2) *admits a unique solution $(p^n,q^n,r^n) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$. Furthermore, the solution (p^n,q^n,r^n) satisfies the following integrability properties, for any $\hat{\varphi} \in \mathcal{A}^U_{\mathbb{E}}(n)$:*

$$E\left[\int_0^T \big(p^n(t)\big)^\mu dt\right] < \infty; \tag{3.3}$$

$$E\left[\int_0^T X_{\hat{\varphi}}(t \wedge \tau_n)^2 \left(\big(q^n(t)\big)^2 + \int_{\mathbb{R}} \big(r^n(t,\zeta)\big)^2 \nu(d\zeta)\right) dt\right] < \infty. \tag{3.4}$$

Proof. See the Appendix. □

Proposition 3.4. *For any fixed $n \in \mathbb{N}$, an element $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves Problem 3.1 if and only if the solution $(p^n,q^n,r^n) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$ to the BSDE* (3.2) *satisfies the following condition:*

$$E\left[b^n(t)\, p^n(t) + \sigma^n(t)\, q^n(t) + \int_{\mathbb{R}} \gamma^n(t,\zeta)\, r^n(t,\zeta)\, \nu(d\zeta) \Big| \mathcal{E}_t\right] = 0 \qquad P\text{-a.s. for a.a. } t \in [0,T]. \tag{3.5}$$

Proof. Let $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ be a solution to Problem 3.1. Since S^{τ_n} is P-a.s. uniformly bounded, it is clear that the strategy $\big\{\beta(t); t \in [0,T]\big\}$ defined by $\beta(t) := \xi \mathbf{1}_{[t_0,t_0+h]}(t)$ belongs to $\mathcal{A}^U_{\mathbb{E}}(n)$, for any $t_0 \in [0,T]$, $h > 0$ and for every bounded $\mathcal{E}_{t_0}$-measurable random variable ξ. Furthermore, since $\eta(t)S(t \wedge \tau_n)$ is P-a.s. uniformly bounded for all $t \in [0,T]$ for any bounded $\eta \in \mathcal{A}^U_{\mathbb{E}}(n)$, it follows that, for every $\psi, \eta \in \mathcal{A}^U_{\mathbb{E}}(n)$ with η bounded, there exists $\bar{\delta} > 0$ such that $\psi + \delta\, \eta \in \mathcal{A}^U_{\mathbb{E}}(n)$ for any $\delta \in \mathbb{R}$ with $|\delta| < \bar{\delta}$. We have thus shown that conditions (A1) and (A2) of the necessary maximum principle of [2] are satisfied. Furthermore, (3.3)-(3.4) together with Remark 2.1 imply that the square-integrability conditions (3.9)-(3.10) of [2] are satisfied up to the stopping time

τ_n, since $(p^n, q^n, r^n) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$. Theorem 3.1 of [2] then implies that:

$$E\left[\frac{\partial H^n}{\partial \varphi}\Big(\omega, t, \varphi(t), p^n(t), q^n(t), r^n(t,\cdot)\Big)\Big|\mathcal{E}_t\right] = 0 \qquad P\text{-a.s. for a.a. } t \in [0,T]. \tag{3.6}$$

Due to equation (3.1), condition (3.6) is easily seen to be equivalent to condition (3.5).
Conversely, suppose that, for some $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$, the unique solution $(p^n, q^n, r^n) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$ to the BSDE (3.2) satisfies condition (3.5). As in the first part of the proof, it is easy to verify that the square-integrability conditions (2.5)-(2.7) of [2] are satisfied (up to τ_n). Furthermore, the random function $\varphi \mapsto H^n\big(\omega, t, \varphi, p^n(t), q^n(t), r^n(t,\cdot)\big)$ is concave in φ and $x \mapsto U(x)$ is concave in x. Since condition (3.5) amounts to the partial information maximum condition (3.6), the partial information sufficient maximum principle given in Theorem 2.1 of [2] implies that φ solves Problem 3.1. □

Condition (3.5) also admits an alternative formulation, in terms of the generalised Malliavin derivatives of the marginal utility U'. To this effect, recall the generalised Clark-Ocone theorem (see [1] for the Brownian motion case and Theorem 3.28 of [8] for the Lévy process case) which states that if the random variable $F \in L^2(P)$ is $\mathcal{F}_T$-measurable, then it can be written as

$$F = E[F] + \int_0^T E[D_t F\,|\mathcal{F}_t]\,dB(t) + \int_0^T\int_{\mathbb{R}} E[D_{t,\zeta} F\,|\mathcal{F}_t]\,\widetilde{N}(dt, d\zeta)$$

where D_t and $D_{t,\zeta}$ denote the generalised Malliavin derivatives at t with respect to B and at t, ζ with respect to N, respectively. Applying this to $F := U'\big(X_\varphi(T\wedge\tau_n)\big)$ we see that the solution (p^n, q^n, r^n) to the BSDE (3.2) can be represented as follows, for all $t \in [0,T]$ and $\zeta \in \mathbb{R}$:

$$\begin{aligned}
p^n(t) &= E\big[U'\big(X_\varphi(T\wedge\tau_n)\big)|\mathcal{F}_t\big],\\
q^n(t) &= E\big[D_t U'\big(X_\varphi(T\wedge\tau_n)\big)|\mathcal{F}_t\big],\\
r^n(t,\zeta) &= E\big[D_{t,\zeta} U'\big(X_\varphi(T\wedge\tau_n)\big)|\mathcal{F}_t\big].
\end{aligned}$$

Therefore, in view of Proposition 3.4, we get the following characterisation of the optimal terminal wealth $X_{\varphi^{*,n}}(T\wedge\tau_n)$ of the partial information locally optimal portfolio problem (Problem 3.1).

Corollary 3.5. *For any fixed $n \in \mathbb{N}$, an element $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves Problem 3.1 if and only if the corresponding terminal wealth $X_\varphi(T\wedge\tau_n)$ satisfies the following partial information Malliavin differential equation P-a.s. for a.a. $t \in [0,T]$:*

$$E\left[b^n(t)\,U'\big(X_\varphi(T\wedge\tau_n)\big) + \sigma^n(t)\,D_t U'\big(X_\varphi(T\wedge\tau_n)\big) + \int_{\mathbb{R}} \gamma^n(t,\zeta)\,D_{t,\zeta} U'\big(X_\varphi(T\wedge\tau_n)\big)\nu(d\zeta)\Big|\mathcal{E}_t\right] = 0.$$

3.2 Partial information equivalent martingale measures (PIEMMs)

We now move to the issue of characterising the density processes of all *partial information equivalent martingale measures (PIEMMs)*, defined below in Definition 3.6. As a preliminary, let us consider a generic probability measure $Q \sim P$ on $(\Omega, \mathcal{F})$ and denote by $G = \big\{G(t); t \in [0,T]\big\}$ its density process, i.e., $G(t) := \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}$ for all $t \in [0,T]$. It is well-known that G is a P-a.s. strictly positive $\mathbb{F}$-martingale with $E[G(T)] = 1$. Furthermore, due to the martingale representation property in the filtration $\mathbb{F}$, there exists an $\mathbb{F}$-predictable process $\theta_0 = \big\{\theta_0(t); t \in [0,T]\big\}$ with $\int_0^T \theta_0^2(t)dt < \infty$ P-a.s. and a $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable function $\theta_1 : \Omega \times [0,T] \times \mathbb{R} \to (-1,\infty)$ with $\int_0^T \int_{\mathbb{R}} \theta_1^2(t,\zeta)\nu(d\zeta)dt < \infty$ P-a.s. such that the following holds:

$$\begin{cases} dG(t) = G(t-)\left(\theta_0(t)\, dB(t) + \displaystyle\int_{\mathbb{R}} \theta_1(t,\zeta)\, \widetilde{N}(dt,d\zeta)\right), & t \in [0,T]; \\ G(0) = 1. \end{cases} \tag{3.7}$$

The SDE (3.7) admits the following explicit solution, for all $t \in [0,T]$:

$$\begin{aligned} G(t) = \exp\bigg(&\int_0^t \theta_0(s)\, dB(s) - \frac{1}{2}\int_0^t \theta_0^2(s)\, ds + \int_0^t\int_{\mathbb{R}} \log\big(1+\theta_1(s,\zeta)\big)\, \widetilde{N}(ds,d\zeta) \\ &+ \int_0^t\int_{\mathbb{R}} \big\{\log\big(1+\theta_1(s,\zeta)\big) - \theta_1(s,\zeta)\big\}\nu(d\zeta)\, ds\bigg). \end{aligned} \tag{3.8}$$

In the following, we write $G_\theta(t) := G(t)$, for $\theta := (\theta_0, \theta_1)$, where $G(t)$ is represented by θ as above. We let Θ denote the family of all $\mathbb{F}$-predictable processes $\theta = (\theta_0,\theta_1)$ such that the SDE (3.7) has a unique strictly positive martingale solution $G_\theta = \big\{G_\theta(t); t \in [0,T]\big\}$. Similarly, for $\theta \in \Theta$, we denote by Q_θ the measure on $(\Omega, \mathcal{F})$ defined by $dQ_\theta/dP := G_\theta(T)$ and by $E_{Q_\theta}[\cdot]$ the corresponding expectation operator. Let us recall that, for any bounded measurable process $Y = \big\{Y(t); t \in [0,T]\big\}$ the $(Q_\theta, \mathbb{E})$-optional projection is the unique $\mathbb{E}$-optional bounded process ${}^oY = \big\{{}^oY(t); t \in [0,T]\big\}$ such that $E_{Q_\theta}\big[Y(\tau)\mathbf{1}_{\{\tau<\infty\}}|\mathcal{E}_\tau\big] = {}^oY(\tau)\mathbf{1}_{\{\tau<\infty\}}$ P-a.s. for every $\mathbb{E}$-stopping time τ (see e.g. [13], Theorem 5.1). In particular, we have ${}^oY(t) = E_{Q_\theta}\big[Y(t)|\mathcal{E}_t\big]$ P-a.s. for every $t \in [0,T]$.

Definition 3.6. *For a fixed $n \in \mathbb{N}$, a probability measure $Q_\theta \sim P$ on $(\Omega, \mathcal{F})$ is said to be a* Partial Information Equivalent Martingale Measure (PIEMM) up to τ_n *if the process ${}^o(S^{\tau_n})$ is a $(Q_\theta, \mathbb{E})$-martingale, where ${}^o(S^{\tau_n})$ denotes the $(Q_\theta, \mathbb{E})$-optional projection of the stopped process S^{τ_n}.*

Remark 3.7. Let us denote by $\mathbb{F}^S = (\mathcal{F}_t^S)_{0\le t\le T}$ the filtration generated by the price process S and suppose that $\mathbb{F}^S \subseteq \mathbb{E}$, as considered e.g. in [4], meaning that all agents have access to the information generated by the observation of the price process S. In this case, if the localizing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ can be chosen to be composed of $\mathbb{F}^S$-stopping times (compare also with Assumption 4.12), a probability measure Q_θ is a PIEMM up to τ_n if and only if the stopped process S^{τ_n} is a $(Q_\theta, \mathbb{E})$-martingale.

In the next proposition we provide a characterisation of the density processes of all PIEMMs.

Proposition 3.8. *For a fixed $n \in \mathbb{N}$, a probability measure $Q_\theta \sim P$ on $(\Omega, \mathcal{F})$ with $(\theta_0, \theta_1) \in$*

$L^2(B) \times G^2(\widetilde{N})$ is a PIEMM up to τ_n if and only if the following condition holds:

$$E_{Q_\theta}\left[b^n(t) + \sigma^n(t)\,\theta_0(t\wedge\tau_n) + \int_{\mathbb{R}} \gamma^n(t,\zeta)\,\theta_1(t\wedge\tau_n,\zeta)\nu(d\zeta)\Big|\mathcal{E}_t\right] = 0 \qquad P\text{-a.s. for a.a. } t \in [0,T]. \tag{3.9}$$

Proof. The process ${}^o(S^{\tau_n})$ is a $(Q_\theta, \mathbb{E})$-martingale if and only if, for every $s,t \in [0,T]$ with $s \le t$, we have $E_{Q_\theta}[\,{}^o(S^{\tau_n})(t)|\mathcal{E}_s] = {}^o(S^{\tau_n})(s)$ P-a.s. By using the conditional version of Bayes' rule and the properties of the $(Q_\theta, \mathbb{E})$-optional projection, we can write:

$$\begin{aligned}
E_{Q_\theta}\left[\,{}^o(S^{\tau_n})(t)|\mathcal{E}_s\right] - {}^o(S^{\tau_n})(s) &= E_{Q_\theta}\left[E_{Q_\theta}[S(t\wedge\tau_n)|\mathcal{E}_t]\big|\mathcal{E}_s\right] - E_{Q_\theta}[S(s\wedge\tau_n)|\mathcal{E}_s] \\
&= E_{Q_\theta}[S(t\wedge\tau_n)|\mathcal{E}_s] - E_{Q_\theta}[S(s\wedge\tau_n)|\mathcal{E}_s] \\
&= \frac{E[G_\theta(t)S(t\wedge\tau_n)|\mathcal{E}_s]}{E[G_\theta(t)|\mathcal{E}_s]} - \frac{E[G_\theta(s)S(s\wedge\tau_n)|\mathcal{E}_s]}{E[G_\theta(s)|\mathcal{E}_s]} \\
&= \frac{E\left[G_\theta(t)S(t\wedge\tau_n) - G_\theta(s)S(s\wedge\tau_n)\big|\mathcal{E}_s\right]}{E[G_\theta(s)|\mathcal{E}_s]}.
\end{aligned} \tag{3.10}$$

Furthermore, by applying the integration by parts formula (see e.g. [20], Lemma 3.6):

$$\begin{aligned}
d\big(G_\theta(t)S(t\wedge\tau_n)\big) &= G_\theta(t-)\,dS(t\wedge\tau_n) + S(t\wedge\tau_n-)\,dG_\theta(t) + d[G_\theta,S](t\wedge\tau_n) \\
&= G_\theta(t-)\,\mathbf{1}_{\{t\le\tau_n\}}\left(b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}}\gamma(t,\zeta)\widetilde{N}(dt,d\zeta)\right) \\
&\quad + S(t\wedge\tau_n-)\left(G_\theta(t-)\Big(\theta_0(t)dB(t) + \int_{\mathbb{R}}\theta_1(t,\zeta)\widetilde{N}(dt,d\zeta)\Big)\right) \\
&\quad + \mathbf{1}_{\{t\le\tau_n\}}G_\theta(t)\sigma(t)\theta_0(t)dt + \mathbf{1}_{\{t\le\tau_n\}}\int_{\mathbb{R}} G_\theta(t-)\gamma(t,\zeta)\theta_1(t,\zeta)N(dt,d\zeta).
\end{aligned}$$

Collecting the dt-terms we get, for all $t \in [0,T]$:

$$G_\theta(t)S(t\wedge\tau_n) = S_0 + \int_0^{t\wedge\tau_n} G_\theta(s)\left(b(s) + \sigma(s)\theta_0(s) + \int_{\mathbb{R}}\gamma(s,\zeta)\theta_1(s,\zeta)\nu(d\zeta)\right)ds + (\mathbb{F}\text{-local martingale}). \tag{3.11}$$

Since $(\theta_0,\theta_1) \in L^2(B)\times G^2(\widetilde{N})$ and S^{τ_n}, b^n, σ^n and γ^n are bounded, it can be easily verified that the $\mathbb{F}$-local martingale term appearing in (3.11) is actually a true $\mathbb{F}$-martingale and, hence, we can write:

$$\begin{aligned}
&E\left[G_\theta(t)S(t\wedge\tau_n) - G_\theta(s)S(s\wedge\tau_n)\big|\mathcal{E}_s\right] = E\Big[E\left[G_\theta(t)S(t\wedge\tau_n) - G_\theta(s)S(s\wedge\tau_n)|\mathcal{F}_s\right]\big|\mathcal{E}_s\Big] \\
&= E\left[\int_{s\wedge\tau_n}^{t\wedge\tau_n} G_\theta(s)\left(b^n(s) + \sigma^n(s)\theta_0(s) + \int_{\mathbb{R}}\gamma^n(s,\zeta)\theta_1(s,\zeta)\nu(d\zeta)\right)ds\Big|\mathcal{E}_s\right].
\end{aligned} \tag{3.12}$$

In view of equation (3.10), this shows that ${}^o(S^{\tau_n})$ is a $(Q_\theta, \mathbb{E})$-martingale if and only if (3.12) is P-a.s. equal to zero for all $s,t \in [0,T]$ with $s \le t$. This is equivalent to the validity of condition (3.9). □

3.3 Local market viability and PIEMMs

We now combine the results of Sections 3.1 and 3.2 to obtain our first main result, namely a characterisation of local market viability in terms of partial information equivalent martingale measures.

Theorem 3.9. *For any fixed $n \in \mathbb{N}$, the following are equivalent:*

- **(i)** *the financial market is locally viable up to τ_n (in the sense of Definition 3.2) and $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves the partial information locally optimal portfolio problem (Problem 3.1);*

- **(ii)** *for $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$, the measure $Q^{\varphi,n} \sim P$ on $(\Omega, \mathcal{F})$ defined by*

$$\frac{dQ^{\varphi,n}}{dP} := \frac{U'\big(X_\varphi(T \wedge \tau_n)\big)}{E\big[U'\big(X_\varphi(T \wedge \tau_n)\big)\big]} \tag{3.13}$$

is a PIEMM up to τ_n, in the sense of Definition 3.6.

Proof. **(i)** $\Rightarrow$ **(ii)**: Due to Proposition 3.4, if $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves Problem 3.1, then the unique solution $(p^n, q^n, r^n) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$ to the BSDE (3.2) satisfies condition (3.5). Let us define, for all $t \in [0,T]$ and $\zeta \in \mathbb{R}$:

$$G(t) := \frac{p^n(t)}{p^n(0)} = \frac{E\big[U'\big(X_\varphi(T \wedge \tau_n)\big)|\mathcal{F}_t\big]}{E\big[U'\big(X_\varphi(T \wedge \tau_n)\big)\big]}, \qquad \theta_0(t) := \frac{q^n(t)}{p^n(t)}, \qquad \theta_1(t,\zeta) := \frac{r^n(t,\zeta)}{p^n(t-)}. \tag{3.14}$$

Then, by combining equation (3.2) with (3.14), we get:

$$\begin{aligned} dG(t) &= \frac{dp^n(t)}{p^n(0)} = \frac{q^n(t)}{p^n(0)}\,dB(t) + \int_{\mathbb{R}} \frac{r^n(t,\zeta)}{p^n(0)}\,\widetilde{N}(dt,d\zeta) = \frac{p^n(t)}{p^n(0)}\,\theta_0(t)\,dB(t) + \frac{p^n(t-)}{p^n(0)}\int_{\mathbb{R}}\theta_1(t,\zeta)\widetilde{N}(dt,d\zeta) \\ &= G(t-)\left[\theta_0(t)\,dB(t) + \int_{\mathbb{R}}\theta_1(t,\zeta)\,\widetilde{N}(dt,d\zeta)\right]. \end{aligned}$$

By letting $dQ^{\varphi,n}/dP := G(T)$, we get a well-defined probability measure $Q^{\varphi,n} \sim P$ on $(\Omega, \mathcal{F})$ with density given by the right-hand side of (3.13). In view of Proposition 3.8, in order to show that $Q^{\varphi,n}$ is a PIEMM up to τ_n it suffices to show that condition (3.9) holds. This follows immediately by substituting (3.14) into condition (3.5).

(ii) $\Rightarrow$ **(i)**: Suppose that the probability measure defined by the right-hand side of (3.13) is a PIEMM up to τ_n, for some $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$, and define the process $G = \big\{G(t); t \in [0,T]\big\}$ as follows:

$$G(t) := \frac{E\big[U'\big(X_\varphi(T \wedge \tau_n)\big)|\mathcal{F}_t\big]}{E\big[U'\big(X_\varphi(T \wedge \tau_n)\big)\big]}, \qquad \text{for all } t \in [0,T].$$

By the martingale representation property, the process G admits a representation of the form (3.7), for some $\mathbb{F}$-predictable process θ_0 and for some $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable function θ_1. Furthermore, since $Q^{\varphi,n}$ is a PIEMM up to τ_n, Proposition 3.8 implies that the following condition is satisfied P-a.s. for a.a. $t \in [0,T]$:

$$E\left[G(t)\left(b^n(t) + \sigma^n(t)\,\theta_0(t\wedge\tau_n) + \int_{\mathbb{R}} \gamma^n(t,\zeta)\,\theta_1(t\wedge\tau_n,\zeta)\nu(d\zeta)\right)\Big|\mathcal{E}_t\right] = 0 \tag{3.15}$$

Let us then define, for all $t \in [0,T]$ and $\zeta \in \mathbb{R}$:

$$p^n(t) := E\big[U'\big(X_\varphi(T\wedge\tau_n)\big)\big]G(t), \qquad q^n(t) := p^n(t)\,\theta_0(t\wedge\tau_n), \qquad r^n(t,\zeta) := p^n(t-)\theta_1(t\wedge\tau_n,\zeta). \tag{3.16}$$

Note that, since the random variable $U'\big(X_\varphi(T\wedge\tau_n)\big) \in L^\mu(P)$, we have $(p^n,q^n,r^n) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$ (compare the proof of Lemma 3.3). By substituting (3.16) into (3.7), we see that (p^n,q^n,r^n) satisfies the BSDE (3.2). Moreover, by substituting (3.16) into (3.15), we can verify that (3.5) holds. Proposition 3.4 allows then to conclude that $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves Problem 3.1. □

Remark 3.10. As we have already mentioned at the end of Section 2, we did not introduce a-priori any no-arbitrage restriction on the financial market. The result of Theorem 3.9 can then be interpreted in the following sense: as soon as the financial market is locally viable, in the sense that a portfolio optimisation problem admits locally a solution, then there exists locally a partial information equivalent martingale measure. This means that the absence of arbitrage opportunities comes as a direct consequence of local market viability. In the papers [17] and [29], the authors develop a general duality theory for the solution of portfolio optimisation problems in the context of financial market models based on general semimartingales. Unlike the present paper, the standing assumption in [17] and [29] is that the set of equivalent (local) martingale measures is non-empty. As we have already pointed out, we opt for a different route and show that the existence of a (partial information) EMM is a consequence of the viability of the financial market (at least locally).

4 Global market viability under partial information

So far, we have studied the viability of the financial market in a local sense, namely up to one of the stopping times composing the localizing sequence $\{\tau_n\}_{n\in\mathbb{N}}$. Now, we adopt a global perspective and aim at characterising the global viability of the financial market. In Section 4.1, we prove a global version of Theorem 3.9 under the stronger assumption that b, σ and γ in (2.1) are bounded (and not only locally bounded). In Section 4.2, we shall deal with the more delicate locally bounded case.

4.1 The case of bounded coefficients

This subsection aims at proving, under a suitable assumption (see Assumption 4.2), the equivalence between the existence of a Partial Information Equivalent Martingale Measure (PIEMM) and the viability of the financial market in a global sense. In the same spirit of Definition 3.2, we can give the following definition of global market viability.

Definition 4.1. *Let U be a utility function. The financial market is said to be* globally viable *if there exists an element $\varphi^* \in \mathcal{A}_{\mathbb{E}}^U$ such that*

$$\sup_{\varphi \in \mathcal{A}_{\mathbb{E}}^U} E\Big[U\big(X_\varphi(T)\big)\Big] = E\Big[U\big(X_{\varphi^*}(T)\big)\Big] < \infty. \tag{4.1}$$

In general, it turns out that the equivalence between the global viability of the financial market (in the sense of Definition 4.1) and the existence of a PIEMM with density given by the (normalised) marginal utility of the optimal terminal wealth does not hold, as shown by an explicit example in Section 5. However, we can still obtain a direct and global version of Theorem 3.9 if the following additional assumption is satisfied.

Assumption 4.2.

(i) The processes b, σ and $\gamma(\cdot, D)$, for every $D \in \mathcal{B}(\mathbb{R})$, are P-a.s. uniformly bounded;

(ii) the utility function U satisfies the following condition:

$$E\Big[U'\big(X_\varphi(T) + \xi\big)^\mu\Big] < \infty, \qquad \text{for all } \varphi \in \mathcal{A}_{\mathbb{E}}^U \text{ and for all } \xi \in \bigcap_{r \in (1,\infty)} L^r(P). \tag{4.2}$$

A useful consequence of Assumption 4.2-(i) is that the price process S admits finite moments of any order. The proof of the next simple lemma is postponed to the Appendix.

Lemma 4.3. *If Assumption 4.2-(i) holds, then $E\big[\sup_{t\in[0,T]} |S(t)|^r\big] < \infty$ for all $r \in (1, \infty)$.*

Following the same approach of Section 3.1, we can characterise the solution to the portfolio optimization problem (4.1) via the solution $(p, q, r) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$ to the associated BSDE

$$\begin{cases} dp(t) = q(t)\, dB(t) + \displaystyle\int_{\mathbb{R}} r(t,\zeta)\, \widetilde{N}(dt, d\zeta), \quad t \in [0,T]; \\ p(T) = U'\big(X_\varphi(T)\big). \end{cases} \tag{4.3}$$

Proposition 4.4. *Suppose that Assumption 4.2 holds. An element $\varphi \in \mathcal{A}_{\mathbb{E}}^U$ solves problem* (4.1) *if and only if the solution $(p, q, r) \in \mathcal{M}^\mu \times L^\mu(B) \times G^\mu(\widetilde{N})$ to the BSDE* (4.3) *satisfies the following condition:*

$$E\bigg[b(t)\, p(t) + \sigma(t)\, q(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\, r(t,\zeta)\, \nu(d\zeta) \bigg| \mathcal{E}_t\bigg] = 0 \qquad P\text{-a.s. for a.a. } t \in [0,T].$$

Proof. Due to Assumption 4.2 together with Lemma 4.3, it is clear that the strategy $\{\beta(t); t \in [0,T]\}$ defined by $\beta(t) := \xi \mathbf{1}_{[t_0, t_0+h]}(t)$ belongs to $\mathcal{A}_{\mathbb{E}}^U$, for any $t_0 \in [0,T]$, $h > 0$ and for every bounded $\mathcal{E}_{t_0}$-measurable random variable ξ. Similarly, for every $\psi, \eta \in \mathcal{A}_{\mathbb{E}}^U$ with η bounded, we have $\psi + \delta\, \eta \in \mathcal{A}_{\mathbb{E}}^U$ for any $\delta \in \mathbb{R}$. This shows that conditions (A1) and (A2) of the necessary maximum principle of [2] are satisfied. By relying on the boundedness of b, σ and γ as well as on Lemma 3.3, the same arguments used in the proof of Proposition 3.4 allow then to prove the claim. □

Recall that, as in Section 3.2, the density process G of any probability measure $Q \sim P$ on $(\Omega, \mathcal{F})$ admits a representation of the form (3.8), for some $\mathbb{F}$-predictable process θ and for some $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable function $\theta_1 : \Omega \times [0,T] \times \mathbb{R} \to (-1, \infty)$. Let us introduce the following definition, which represents a natural extension of Definition 3.6 to the global case.

Definition 4.5. *A probability measure $Q_\theta \sim P$ on $(\Omega, \mathcal{F})$ is said to be a* Partial Information Equivalent Martingale Measure (PIEMM) *if the process ${}^oS$ is a $(Q_\theta, \mathbb{E})$-martingale, where ${}^oS$ denotes the $(Q_\theta, \mathbb{E})$-optional projection of the process S.*

Density processes G_θ of PIEMMs can be characterised as follows, similarly to Proposition 3.8.

Proposition 4.6. *Suppose that Assumption 4.2-(i) holds. A probability measure $Q_\theta \sim P$ on $(\Omega, \mathcal{F})$ with $dQ_\theta/dP \in L^\mu(P)$ is a PIEMM if and only if the following condition holds:*

$$E_{Q_\theta}\left[b(t) + \sigma(t)\,\theta_0(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\,\theta_1(t,\zeta)\nu(d\zeta)\Big|\mathcal{E}_t\right] = 0 \qquad P\text{-a.s. for a.a. } t \in [0,T]$$

where θ_0 (θ_1, resp.) is the process (predictable function, resp.) appearing in the representation (3.8).

Proof. The claim can be proved by relying on the same arguments used in the proof of Proposition 3.8 (of course, without stopping by τ_n). Note that, in the present context, the true $\mathbb{F}$-martingale property of the $\mathbb{F}$-local martingale part in (3.11) follows from Lemma 4.3 together with a simple application of the Burkholder-Davis-Gundy and Doob's inequalities, using that $dQ_\theta/dP \in L^\mu(P)$. □

As in Section 3.3, we can now combine Propositions 4.4 and 4.6 in order to obtain the equivalence between the global viability of the financial market (in the sense of Definition 4.1) and the existence of a PIEMM. We omit the proof, which is entirely similar to the proof of Theorem 3.9.

Theorem 4.7. *Suppose that Assumption 4.2 holds. Then the following are equivalent:*

- **(i)** *the financial market is globally viable, in the sense of Definition 4.1, and $\varphi \in \mathcal{A}^U_{\mathbb{E}}$ solves the partial information optimal portfolio problem* (4.1)*;*
- **(ii)** *for $\varphi \in \mathcal{A}^U_{\mathbb{E}}$, the measure $Q^\varphi \sim P$ on $(\Omega, \mathcal{F})$ defined by*

$$\frac{dQ^\varphi}{dP} := \frac{U'\big(X_\varphi(T)\big)}{E\big[U'\big(X_\varphi(T)\big)\big]}$$

is a PIEMM, in the sense of Definition 3.8.

Remark 4.8. Of course, Assumption 4.2 is satisfied for any discrete-time financial market model on a finite probability space. In that case, Theorem 4.7 allows to recover a classical and well-known result from financial economics, as discussed in Section 4.4 of [3] (see also [22] and [23]).

4.2 The general case

In the present section, we study the issue of global market viability in the more general case of locally bounded coefficients b, σ and γ, as in Section 2, without assuming that the simplifying Assumption 4.2 is satisfied. In this case, as will be shown by the explicit example contained in Section 5, we cannot characterise global market viability in terms of PIEMMs and we need to rely on the localization approach described in Section 3, adopting the following definition of global market viability.

Definition 4.9 (Global market viability)**.** *Let U be a utility function. The financial market is said to be* globally viable *if Problem 3.1 admits an optimal solution $\varphi^* \in \mathcal{A}^U_{\mathbb{E}}(n)$ for all $n \in \mathbb{N}$ and if the family of processes $\{p^n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}^\mu$ solving the associated BSDEs* (3.2) *satisfies the following consistency condition:*

$$E\left[p^n(t)/p^n(0)\,|\,\mathcal{F}_{t\wedge\tau_{n-1}}\right] = p^{n-1}(t)/p^{n-1}(0), \qquad \text{for all } t \in [0,T] \text{ and } n \in \mathbb{N}. \tag{4.4}$$

In general (i.e., for unbounded coefficients), one cannot hope to obtain a full characterisation of global market viability in terms of (partial information) equivalent martingale measures (not even in terms of partial information equivalent local martingale measures), as will be shown in Section 5. Hence, we need to introduce the following concept, which corresponds to a weaker counterpart of the density process of a PIEMM and extends to the partial information setting the notion of local martingale deflator introduced in [16].

Definition 4.10. *A strictly positive $\mathbb{F}$-local martingale $Z = \{Z(t); t \in [0,T]\}$ with $Z_0 = 1$ is said to be a* Partial Information Local Martingale Deflator (PILMD) *if the product ${}^oZ\,{}^oS$ is an $\mathbb{E}$-local martingale, with ^{o} denoting the $(P,\mathbb{E})$-optional projection.*

Remark 4.11. In the complete information case (i.e., $\mathbb{E} = \mathbb{F}$), it has been recently shown in [16] that the existence of a local martingale deflator is equivalent to the *No Arbitrage of the First Kind (NA1)* condition, which is in turn equivalent to the absence of *Unbounded Profits with Bounded Risk (NUPBR)*, as considered in [15]. In particular, these two no-arbitrage conditions can be shown to be strictly weaker than the classical *No Free Lunch with Vanishing Risk (NFLVR)* condition introduced in [5], the latter being equivalent to the existence of an Equivalent Local Martingale Measure. In Section 5 we will propose an example of a globally viable financial market which does not satisfy NFLVR.

Until the end of this section, we shall work under the following technical assumption.

Assumption 4.12. The localizing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ can be chosen to be composed of $\mathbb{E}$-stopping times.

Remarks 4.13. 1) The reason why we require Assumption 4.12 consists in the fact that we shall need to take $\mathbb{E}$-optional projections of $\mathbb{F}$-local martingales. It is well-known (and easy to check) that the $\mathbb{E}$-optional projection of an $\mathbb{F}$-martingale is an $\mathbb{E}$-martingale. However, when one considers an $\mathbb{F}$-local martingale, its $\mathbb{E}$-optional projection can fail to be an $\mathbb{E}$-local martingale. As can be easily verified, a sufficient condition for the $\mathbb{E}$-optional projection ${}^oY$ of an $\mathbb{F}$-local martingale Y to be an $\mathbb{E}$-local martingale is that Y can be localized with a sequence of $\mathbb{E}$-stopping times (see also [10], Theorem 3.7).

2) Note that, as long as Assumption 4.12 holds, we have ${}^o(S^{\tau_n}) = {}^oS^{\tau_n}$ for every $n \in \mathbb{N}$, since the process $\mathbf{1}_{[\![0,\tau_n]\!]}$ is $\mathbb{E}$-optional whenever τ_n is an $\mathbb{E}$-stopping time (compare also with [13], Theorem 5.7).

By relying on Theorem 3.9, we are now in a position to formulate the next theorem, which gives a characterisation of global market viability (in the sense of Definition 4.9) under partial information.

Theorem 4.14. *Suppose that Assumption 4.12 holds. The following are equivalent:*

(i) *the financial market is globally viable, in the sense of Definition 4.9, and $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves the partial information locally optimal portfolio problem (Problem 3.1) for all $n \in \mathbb{N}$;*

(ii) *for some $\varphi \in \bigcap_{n\in\mathbb{N}} \mathcal{A}^U_{\mathbb{E}}(n)$, the process $Z_\varphi = \{Z_\varphi(t) : t \in [0,T]\}$ defined by*

$$Z_\varphi(t) := \mathbf{1}_{\{t=0\}} + \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}} \frac{E\big[U'\big(X_\varphi(T\wedge\tau_k)\big)|\mathcal{F}_t\big]}{E\big[U'\big(X_\varphi(T\wedge\tau_k)\big)\big]}, \qquad \text{for all } t \in [0,T], \tag{4.5}$$

is a PILMD satisfying $Z_\varphi(T\wedge\tau_n) = U'\big(X_\varphi(T\wedge\tau_n)\big)/E\big[U'\big(X_\varphi(T\wedge\tau_n)\big)\big]$, for all $n \in \mathbb{N}$, with $\tau_0 := 0$.

Proof. **(i) ⇒ (ii)**: Suppose that, for all $n \in \mathbb{N}$, the strategy $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves Problem 3.1 and define the process $Z^k_\varphi = \{Z^k_\varphi(t); t \in [0,T]\}$ as follows, for every $k \in \mathbb{N}$:

$$Z^k_\varphi(t) := \frac{E\big[U'\big(X_\varphi(T\wedge\tau_k)\big)|\mathcal{F}_t\big]}{E\big[U'\big(X_\varphi(T\wedge\tau_k)\big)\big]}, \qquad \text{for all } t \in [0,T], \tag{4.6}$$

Then, let us define the process $Z_\varphi = \{Z_\varphi(t); t \in [0,T]\}$ by $Z_\varphi(t) := 1 + \sum_{k=1}^{\infty} \int_{t\wedge\tau_{k-1}}^{t\wedge\tau_k} dZ^k_\varphi(s)$, for all $t \in [0,T]$. Since $Z^k_\varphi \in \mathcal{M}^\mu$, for all $k \in \mathbb{N}$, the process Z_φ is an $\mathbb{F}$-local martingale with $Z_\varphi(0) = 1$ and, furthermore, it is localized by the sequence $\{\tau_n\}_{n\in\mathbb{N}}$. As can be easily checked, the consistency condition (4.4) implies that $Z^k_\varphi(t\wedge\tau_{k-1}) = Z^{k-1}_\varphi(t\wedge\tau_{k-1})$ for all $t \in [0,T]$ and $k \in \mathbb{N}$, so that the process Z_φ can be equivalently rewritten as follows, for all $t \in [0,T]$:

$$Z_\varphi(t) = \mathbf{1}_{\{t=0\}} + \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}} Z^k_\varphi(t) = \mathbf{1}_{\{t=0\}} + \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}} \frac{E\big[U'\big(X_\varphi(T\wedge\tau_k)\big)|\mathcal{F}_t\big]}{E\big[U'\big(X_\varphi(T\wedge\tau_k)\big)\big]}.$$

In view of Definition 4.10, it remains to show that ${}^oZ\,{}^oS$ is an $\mathbb{E}$-local martingale. Recall that, due to Assumption 4.12, we can interchange the operations of taking the $\mathbb{E}$-optional projection and stopping by τ_n, for all $n \in \mathbb{N}$. Hence, we can write, for all $t \in [0,T]$ and $n \in \mathbb{N}$:

$$
\begin{aligned}
{}^{o}Z_{\varphi}(t\wedge\tau_n)\,{}^{o}S(t\wedge\tau_n) &= S_0\mathbf{1}_{\{t=0\}} + {}^{o}S(t\wedge\tau_n)\left(\sum_{k=1}^{n}\mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}}\,{}^{o}Z^{k}_{\varphi}(t) + \mathbf{1}_{\{\tau_n<t\}}\,{}^{o}Z^{n}_{\varphi}(\tau_n)\right)\\
&= S_0\mathbf{1}_{\{t=0\}} + \sum_{k=1}^{n}\mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}}\,{}^{o}Z^{k}_{\varphi}(t)\,{}^{o}S(t) + \mathbf{1}_{\{\tau_n<t\}}\,{}^{o}Z^{n}_{\varphi}(\tau_n)\,{}^{o}S(\tau_n)\\
&= S_0\mathbf{1}_{\{t=0\}} + \sum_{k=1}^{n}\mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}}\,{}^{o}Z^{k}_{\varphi}(t)\,{}^{o}(S^{\tau_k})(t) + \mathbf{1}_{\{\tau_n<t\}}\,{}^{o}Z^{n}_{\varphi}(\tau_n)\,{}^{o}(S^{\tau_n})(\tau_n)\\
&= S_0 + \sum_{k=1}^{n}\int_{t\wedge\tau_{k-1}}^{t\wedge\tau_k} d\big({}^{o}Z^{k}_{\varphi}\,{}^{o}(S^{\tau_k})\big)(s).
\end{aligned}
\tag{4.7}
$$

Due to Theorem 3.9, the process Z^k_{φ} is the density process of a PIEMM up to τ_k, for every $k\in\mathbb{N}$, thus implying that ${}^{o}Z^{k}_{\varphi}\,{}^{o}(S^{\tau_k})$ is an $\mathbb{E}$-martingale, for every $k\in\mathbb{N}$. Together with equation (4.7), this shows that $({}^{o}Z_{\varphi}\,{}^{o}S)^{\tau_n}$ is an $\mathbb{E}$-martingale for all $n\in\mathbb{N}$, or, equivalently, that ${}^{o}Z_{\varphi}\,{}^{o}S$ is an $\mathbb{E}$-local martingale, thus proving that Z_{φ} is a PILMD. To complete the proof of the implication (i) $\Rightarrow$ (ii), note that the consistency condition (4.4) implies that, for all $n\in\mathbb{N}$:

$$
\begin{aligned}
Z_{\varphi}(T\wedge\tau_n) &= \sum_{k=1}^{n}\mathbf{1}_{\{\tau_{k-1}<T\leq\tau_k\}}\frac{U'\big(X_{\varphi}(T\wedge\tau_k)\big)}{E\big[U'\big(X_{\varphi}(T\wedge\tau_k)\big)\big]} + \mathbf{1}_{\{T>\tau_n\}}\frac{U'\big(X_{\varphi}(T\wedge\tau_n)\big)}{E\big[U'\big(X_{\varphi}(T\wedge\tau_n)\big)\big]}\\
&= \sum_{k=1}^{n}\mathbf{1}_{\{\tau_{k-1}<T\leq\tau_k\}}\frac{p^k(T)}{p^k(0)} + \mathbf{1}_{\{T>\tau_n\}}\frac{p^n(T)}{p^n(0)}\\
&= \sum_{k=1}^{n}\mathbf{1}_{\{\tau_{k-1}<T\leq\tau_k\}}E\left[\frac{p^n(T)}{p^n(0)}\Big|\mathcal{F}_{T\wedge\tau_k}\right] + \mathbf{1}_{\{T>\tau_n\}}\frac{p^n(T)}{p^n(0)} = \frac{p^n(T)}{p^n(0)} = \frac{U'\big(X_{\varphi}(T\wedge\tau_n)\big)}{E\big[U'\big(X_{\varphi}(T\wedge\tau_n)\big)\big]}
\end{aligned}
$$

(ii) $\Rightarrow$ (i): Suppose that, for some $\varphi\in\bigcap_{n\in\mathbb{N}}\mathcal{A}^{U}_{\mathbb{E}}(n)$, the process Z_{φ} defined in (4.5) is a PILMD. Due to the local boundedness assumption (see Remark 2.1), the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is a localizing sequence for Z_{φ}, meaning that the stopped process $Z^{\tau_n}_{\varphi}$ is an $\mathbb{F}$-martingale, for every $n\in\mathbb{N}$. Moreover, since Z_{φ} is a PILMD, the stopped process $\big({}^{o}Z_{\varphi}\,{}^{o}S\big)^{\tau_n} = {}^{o}\big(Z^{\tau_n}_{\varphi}\big)\,{}^{o}\big(S^{\tau_n}\big)$ is an $\mathbb{E}$-martingale, for every $n\in\mathbb{N}$, where we have also used Assumption 4.12. In view of Definition 3.6, this means that the measure Q_{φ} defined by $dQ_{\varphi}:=Z_{\varphi}(T\wedge\tau_n)dP$ defines a PIEMM up to τ_n. Theorem 3.9 then implies that φ solves Problem 3.1, for all $n\in\mathbb{N}$. To complete the proof, it remains to prove the consistency condition (4.4). This can be shown as follows, using the $\mathbb{F}$-martingale property of p^n and $Z^{\tau_n}_{\varphi}$, for all $n\in\mathbb{N}$ and $t\in[0,T]$:

$$
\begin{aligned}
E\left[\frac{p^n(t)}{p^n(0)}\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\right]-\frac{p^{n-1}(t)}{p^{n-1}(0)} &= E\left[\frac{p^n(T)}{p^n(0)}-\frac{p^{n-1}(T)}{p^{n-1}(0)}\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\right]\\
&= E\left[\frac{U'\big(X_\varphi(T\wedge\tau_n)\big)}{E\big[U'\big(X_\varphi(T\wedge\tau_n)\big)\big]}-\frac{U'\big(X_\varphi(T\wedge\tau_{n-1})\big)}{E\big[U'\big(X_\varphi(T\wedge\tau_{n-1})\big)\big]}\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\right]\\
&= E\big[Z_\varphi(T\wedge\tau_n)-Z_\varphi(T\wedge\tau_{n-1})\big|\mathcal{F}_{t\wedge\tau_{n-1}}\big]\\
&= E\Big[E\big[Z_\varphi(T\wedge\tau_n)-Z_\varphi(T\wedge\tau_{n-1})\big|\mathcal{F}_{T\wedge\tau_{n-1}}\big]\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\Big]=0
\end{aligned}
$$

In view of Definition 4.9, we have thus shown that the financial market is globally viable. □

In particular, we want to remark that Theorem 4.14 not only shows that if the financial market is globally viable (in the sense of Definition 4.9) then there exists a partial information local martingale deflator, but also gives an explicit description of the PILMD Z_φ, which aggregates the expected (normalised) marginal utilities of terminal wealth at the stopping times of the localizing sequence $\{\tau_n\}_{n\in\mathbb{N}}$.

Remark 4.15 (*On the case of bounded coefficients*)**.** If we suppose, as in Section 4.1, that Assumption 4.2 is satisfied, then the PILMD Z_φ appearing in (4.5) is in fact the density process of the PIEMM Q_φ defined in part (ii) of Theorem 4.7. Indeed, Assumption 4.2 implies that there exists an element $n^*\in\mathbb{N}$ such that $\tau_{n^*}=\infty$ P-a.s. This means that Z_φ in (4.5) reduces to the finite sum of the first n^* terms, thus implying that Z_φ is a true $\mathbb{F}$-martingale. Furthermore, it is easy to check that the consistency condition (4.4) implies that, for all $k=0,1,\ldots,n^*-1$:

$$
\begin{aligned}
\frac{U'\big(X_\varphi(T\wedge\tau_k)\big)}{E\big[U'\big(X_\varphi(T\wedge\tau_k)\big)\big]} &= \frac{p^k(T)}{p^k(0)} = E\left[\frac{p^{n^*}(T)}{p^{n^*}(0)}\Big|\mathcal{F}_{T\wedge\tau_k}\right] = E\left[\frac{U'\big(X_\varphi(T\wedge\tau_{n^*})\big)}{E\big[U'\big(X_\varphi(T\wedge\tau_{n^*})\big)\big]}\Big|\mathcal{F}_{T\wedge\tau_k}\right]\\
&= E\left[\frac{U'\big(X_\varphi(T)\big)}{E\big[U'\big(X_\varphi(T)\big)\big]}\Big|\mathcal{F}_{T\wedge\tau_k}\right].
\end{aligned}
$$

By substitution into equation (4.5) we get:

$$
Z_\varphi(T)=\sum_{k=1}^{n^*}\mathbf{1}_{\{\tau_{k-1}<T\le\tau_k\}}\,E\left[\frac{U'\big(X_\varphi(T)\big)}{E\big[U'\big(X_\varphi(T)\big)\big]}\Big|\mathcal{F}_{T\wedge\tau_k}\right]=\frac{U'\big(X_\varphi(T)\big)}{E\big[U'\big(X_\varphi(T)\big)\big]}
$$

thus confirming the result of Theorem 4.7.

5 An example

This section is meant to be an illustration of the concepts discussed so far in context of a simple continuous financial market model, based on a three-dimensional Bessel process. Bessel processes have been extensively studied in relation with the existence of arbitrage opportunities, see e.g. [6], Section 2 of [11] and Example 4.6 in [15]. Nevertheless, in the context of the present example, we will show that the financial market is viable in the local as well as in the global sense for a logarithmic utility function, even though the model allows for classical arbitrage opportunities.

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a given filtered probability space, with a standard Brownian motion B and where $\mathbb{E} := \mathbb{F}^B \subseteq \mathbb{F}$ is the P-augmented filtration generated by B. We define the discounted price process S of a single risky asset as the solution to the following SDE:

$$\begin{cases} dS(t) = \dfrac{1}{S(t)}\,dt + dB(t), & t \in [0,T]; \\ S(0) = 1. \end{cases} \tag{5.1}$$

The solution to the SDE (5.1) is a P-a.s. strictly positive process known as the three-dimensional Bessel process (we refer the reader to Chapter XI of [27] for a detailed study of Bessel processes). It is easy to see that there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of $\mathbb{E}$-stopping times with $\tau_n \nearrow +\infty$ P-a.s. as $n \to +\infty$ such that S^{τ_n} and $1/S^{\tau_n}$ are P-a.s. uniformly bounded, for every $n \in \mathbb{N}$. Indeed, it suffices to define $\tau_n := \inf\bigl\{t \in [0,T] : S(t) \notin (1/n, n)\bigr\}$, for $n \in \mathbb{N}$ (with the usual convention $\inf\emptyset = +\infty$). A simple application of Itô's formula gives that $dS^{-1}(t) = -S^{-2}(t)\,dB(t)$, thus showing that $1/S$ is an $\mathbb{E}$-local martingale or, equivalently, that the stopped process $1/S^{\tau_n}$ is an $\mathbb{E}$-martingale, for all $n \in \mathbb{N}$. Furthermore, for any $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$ and for any $n \in \mathbb{N}$:

$$\begin{aligned} d\left(\frac{X_\varphi(t\wedge\tau_n)}{S(t\wedge\tau_n)}\right) &= X_\varphi(t\wedge\tau_n)\,d\frac{1}{S(t\wedge\tau_n)} + \frac{\varphi(t)}{S(t\wedge\tau_n)}\,dS(t\wedge\tau_n) - \frac{\varphi(t)}{S^2(t\wedge\tau_n)}dt \\ &= X_\varphi(t\wedge\tau_n)\,d\frac{1}{S(t\wedge\tau_n)} + \frac{\varphi(t)}{S(t\wedge\tau_n)}\,dB(t) \end{aligned} \tag{5.2}$$

thus showing that the stopped process $X^{\tau_n}_\varphi/S^{\tau_n}$ is an $\mathbb{E}$-martingale, for every $n \in \mathbb{N}$.

Let us consider the logarithmic utility function $U(x) = \log(x)$, with an initial endowment of $x = 1$. Jensen's inequality together with the martingale property of $X^{\tau_n}_\varphi/S^{\tau_n}$ gives

$$E\Bigl[\log\bigl(X_\varphi(T\wedge\tau_n)/S(T\wedge\tau_n)\bigr)\Bigr] \leq \log\Bigl(E\bigl[X_\varphi(T\wedge\tau_n)/S(T\wedge\tau_n)\bigr]\Bigr) = 0$$

meaning that $E\bigl[\log\bigl(X_\varphi(T\wedge\tau_n)\bigr)\bigr] \leq E\bigl[\log\bigl(S(T\wedge\tau_n)\bigr)\bigr]$, for any $\varphi \in \mathcal{A}^U_{\mathbb{E}}(n)$. This shows that the optimal strategy for the logarithmic utility consists in a simple buy-and-hold position in the risky asset itself, i.e., $\varphi^{*,n} = 1 \in \mathcal{A}^U_{\mathbb{E}}(n)$ for all $n \in \mathbb{N}$. According to Definition 3.2, the financial market is locally viable up to τ_n.

We can also verify the local viability of the financial market by applying Theorem 3.9. Indeed, since the stopped process $1/S^{\tau_n}$ is a strictly positive $\mathbb{E}$-martingale, we can define a probability measure Q^n on $(\Omega, \mathcal{F})$ by letting $dQ^n/dP := 1/S(T\wedge\tau_n)$. Due to Bayes' rule, it is easy to check that Q^n is a PIEMM up to τ_n, in the sense of Definition 3.6 (noting that we are in the situation described in Remark 3.7). Since $U'(x) = 1/x$, the implication (ii) $\Rightarrow$ (i) of Theorem 3.9 implies that the financial market is viable up to τ_n and that $\varphi^{*,n} = 1 \in \mathcal{A}^U_{\mathbb{E}}(n)$ solves Problem 3.1.

Since the process $1/S$ is unbounded, Assumption 4.2 fails to hold and, hence, we cannot rely on the approach presented in Section 4.1 to study the global viability of the financial market.

Moreover, we can prove that the process $1/S$ cannot be used as the density process of a PIEMM on $[0,T]$, since $1/S$ is a *strict* local martingale, according to the terminology of [9], being a local martingale which fails to be a true martingale, so that $E[1/S(T)]<1$. Let us explain with some more details this phenomenon. We define the measure Q^{φ^*} as follows:

$$\frac{dQ^{\varphi^*}}{dP}:=\frac{U'\big(X_{\varphi^*}(T)\big)}{E\big[U'\big(X_{\varphi^*}(T)\big)\big]}=\frac{1/S(T)}{E\big[1/S(T)\big]}.$$

If Q^{φ^*} were a PIEMM then its density process $G=\big\{G(t);t\in[0,T]\big\}$, with $dQ^{\varphi^*}|_{\mathcal{E}_t}:=G(t)\,dP|_{\mathcal{E}_t}$ for all $t\in[0,T]$, would be an $\mathbb{E}$-martingale admitting the following representation, as in (3.8):

$$G(t)=\exp\left(\int_0^t\theta_0(s)dB(s)-\frac{1}{2}\int_0^t\theta_0^2(s)ds\right),\qquad\text{for all }t\in[0,T],$$

for some $\mathbb{E}$-predictable process $\theta_0=\big\{\theta_0(t);t\in[0,T]\big\}$ with $\int_0^T\theta_0^2(t)dt<\infty$ P-a.s., so that:

$$d\big(G(t)S(t)\big)=S(t)dG(t)+G(t)dS(t)+d\langle G,S\rangle(t)=S(t)dG(t)+G(t)dB(t)+G(t)\left(\frac{1}{S(t)}+\theta_0(t)\right)dt. \tag{5.3}$$

If Q^{φ^*} were a PIEMM, then the product GS would be an $\mathbb{E}$-(local) martingale and equation (5.3) would then imply that $\theta_0(t)=-1/S(t)$ for a.a. $t\in[0,T]$, thus implying that $dG(t)=-G(t)/S(t)\,dB(t)$. But, since $G(0)=1/S(0)=1$, this would in turn imply that G and $1/S$ solve the same SDE and, hence, one would conclude $G=1/S$, thus contradicting the martingale property of G. This shows that, in the context of the present example, the marginal utility of the optimal terminal wealth cannot be taken as the density of a PIEMM. The failure of the martingale property of $1/S$ is also linked to the existence of multiple solutions to the BSDE (3.2) on the time horizon $[0,T]$ beyond the class $\mathcal{M}^2\times L^2(B)$, as discussed in the recent paper [32].

We conclude the discussion of this example by showing that the financial market is globally viable, in the sense of Definition 4.9. Indeed, we already know that $\varphi^*:=1=\varphi^{*,n}\in\mathcal{A}^U_{\mathbb{E}}(n)$ solves Problem 3.1 for the logarithmic utility for all $n\in\mathbb{N}$. Moreover, the consistency condition (4.4) also holds, due to the martingale property of the stopped process $1/S^{\tau_n}$, for all $n\in\mathbb{N}$ and $t\in[0,T]$:

$$\begin{aligned}E\left[\frac{p^n(t)}{p^n(0)}-\frac{p^{n-1}(t)}{p^{n-1}(0)}\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\right]&=E\left[\frac{E\big[U'\big(X_{\varphi^*}(T\wedge\tau_n)\big)|\mathcal{F}_t\big]}{E\big[U'\big(X_{\varphi^*}(T\wedge\tau_n)\big)\big]}-\frac{E\big[U'\big(X_{\varphi^*}(T\wedge\tau_{n-1})\big)|\mathcal{F}_t\big]}{E\big[U'\big(X_{\varphi^*}(T\wedge\tau_{n-1})\big)\big]}\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\right]\\&=E\left[\frac{E\big[1/S(T\wedge\tau_n)|\mathcal{F}_t\big]}{E\big[1/S(T\wedge\tau_n)\big]}-\frac{E\big[1/S(T\wedge\tau_{n-1})|\mathcal{F}_t\big]}{E\big[1/S(T\wedge\tau_{n-1})\big]}\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\right]\\&=E\left[\frac{1}{S(t\wedge\tau_n)}-\frac{1}{S(t\wedge\tau_{n-1})}\Big|\mathcal{F}_{t\wedge\tau_{n-1}}\right]=0.\end{aligned}$$

Alternatively, we can prove the global viability of the financial market by applying Theorem 4.14. Indeed, let us take $\varphi=1\in\bigcap_{n\in\mathbb{N}}\mathcal{A}^U_{\mathbb{E}}(n)$ and consider the process Z_φ defined in (4.5).

Since $U'(x)=1/x$, for every $n\in\mathbb{N}$, it is immediate to check that $Z_\varphi=1/S$:

$$Z_\varphi(t)=\mathbf{1}_{\{t=0\}}+\sum_{k=1}^{\infty}\mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}}\frac{E\big[1/S(T\wedge\tau_k)|\mathcal{F}_t\big]}{E\big[1/S(T\wedge\tau_k)\big]}=\mathbf{1}_{\{t=0\}}+\sum_{k=1}^{\infty}\mathbf{1}_{\{\tau_{k-1}<t\leq\tau_k\}}\frac{1}{S(t\wedge\tau_k)}=\frac{1}{S(t)}.$$

Equation (5.2) together with the martingale property of $1/S^{\tau_n}$, for all $n\in\mathbb{N}$, shows that $1/S$ is a PILMD and Theorem 4.14 implies then that the financial market is globally viable.

A Appendix

Proof of Lemma 3.3.

The existence of a unique solution $(p^n,q^n,r^n)\in\mathcal{M}^\mu\times L^\mu(B)\times G^\mu(\widetilde{N})$ can be shown by applying Proposition A.5 of [25], but we prefer to give full details for the convenience of the reader. For any fixed $\varphi\in\mathcal{A}^U_{\mathbb{E}}(n)$, we have $U'\big(X_\varphi(T\wedge\tau_n)\big)\in L^\mu(P)$ and, hence, the $\mathbb{F}$-martingale $p^n=\big\{p^n(t);t\in[0,T]\big\}$ defined by $p^n(t):=E\big[U'\big(X_\varphi(T\wedge\tau_n)\big)|\mathcal{F}_t\big]$, for all $t\in[0,T]$, satisfies $p^n(T)=U'\big(X_\varphi(T\wedge\tau_n)\big)$ and belongs to $\mathcal{M}^\mu$, as a consequence of Doob's inequality. Since the pair $(B,\widetilde{N})$ enjoys the martingale representation property in the filtration $\mathbb{F}$ (see e.g. [28], Theorem 2.3) and since the martingale representation property is stable under stopping (see e.g. [13], Lemma 13.8), there exists a unique couple $(q^n,r^n)\in L^2(B)\times G^2(\widetilde{N})$ such that (3.2) is satisfied. The fact that $(q^n,r^n)\in L^\mu(B)\times G^\mu(\widetilde{N})$ follows from the Burkholder-Davis-Gundy and Doob's inequalities, since $p^n(T)\in L^\mu(P)$.

It remains to prove the integrability properties (3.3)-(3.4). For (3.3), it suffices to note that, using Doob's inequality:

$$E\left[\int_0^T\big(p^n(t)\big)^\mu dt\right]\leq TE\left[\sup_{t\in[0,T]}\big|p^n(t)\big|^\mu\right]\leq C_\mu TE\left[\big(p^n(T)\big)^\mu\right]=C_\mu TE\Big[U'\big(X_\varphi(T\wedge\tau_n)\big)^\mu\Big]<\infty.$$

To prove (3.4), we can argue as follows, using in sequence Hölder's inequality, the inequality between the predictable and the optional quadratic variation given in Corollary 1.3 of [24], Burkholder-Davis-Gundy's inequality and Doob's inequality and where C represents a positive constant which can change from line to line:

$$
\begin{aligned}
& E\left[\int_0^T X_{\hat\varphi}(t\wedge\tau_n)^2\left(\big(q^n(t)\big)^2+\int_{\mathbb{R}}\big(r^n(t,\zeta)\big)^2\nu(d\zeta)\right)dt\right]\\
\le\ & E\Big[\sup_{t\in[0,T\wedge\tau_n]}\big|X_{\hat\varphi}(t)\big|^\lambda\Big]^{\frac{2}{\lambda}} E\left[\left(\int_0^T\big(q^n(t)\big)^2+\int_0^T\int_{\mathbb{R}}\big(r^n(t,\zeta)\big)^2\nu(d\zeta)\,dt\right)^{\frac{\mu}{2}}\right]^{\frac{2}{\mu}}\\
=\ & E\Big[\sup_{t\in[0,T\wedge\tau_n]}\big|X_{\hat\varphi}(t)\big|^\lambda\Big]^{\frac{2}{\lambda}} E\Big[\Big\langle\int q^n dB+\int\int_{\mathbb{R}}r^n d\widetilde N\Big\rangle_T^{\frac{\mu}{2}}\Big]^{\frac{2}{\mu}}\\
\le\ & CE\Big[\sup_{t\in[0,T\wedge\tau_n]}\big|X_{\hat\varphi}(t)\big|^\lambda\Big]^{\frac{2}{\lambda}} E\Big[\Big[\int q^n dB+\int\int_{\mathbb{R}}r^n d\widetilde N\Big]_T^{\frac{\mu}{2}}\Big]^{\frac{2}{\mu}}\\
\le\ & CE\Big[\sup_{t\in[0,T\wedge\tau_n]}\big|X_{\hat\varphi}(t)\big|^\lambda\Big]^{\frac{2}{\lambda}} E\left[\sup_{t\in[0,T]}\left|\int_0^t q^n(u)dB(u)\int_0^t\int_{\mathbb{R}}r^n(u,\zeta)\widetilde N(du,d\zeta)\right|^\mu\right]^{\frac{2}{\mu}}\\
\le\ & CE\Big[\sup_{t\in[0,T\wedge\tau_n]}\big|X_{\hat\varphi}(t)\big|^\lambda\Big]^{\frac{2}{\lambda}} E\left[\left(\int_0^T q^n(u)dB(u)+\int_0^T\int_{\mathbb{R}}r^n(u,\zeta)\widetilde N(du,d\zeta)\right)^\mu\right]^{\frac{2}{\mu}}\\
=\ & CE\Big[\sup_{t\in[0,T\wedge\tau_n]}\big|X_{\hat\varphi}(t)\big|^\lambda\Big]^{\frac{2}{\lambda}} E\Big[\Big(U'\big(X_\varphi(T\wedge\tau_n)\big)-E\big[U'\big(X_\varphi(T\wedge\tau_n)\big)\big]\Big)^\mu\Big]^{\frac{2}{\mu}}<\infty
\end{aligned}
$$

where the finiteness of the last expectations follows from the admissibility $\varphi,\hat\varphi\in\mathcal{A}^U_{\mathbb{E}}(n)$.

Proof of Lemma 4.3.

We use Minkowski's inequality and the Burkholder-Davis-Gundy inequality to get, for any $r\in(1,\infty)$:

$$
\begin{aligned}
& E\left[\sup_{t\in[0,T]}\left|\int_0^t\sigma(u)dB(u)+\int_0^t\int_{\mathbb{R}}\gamma(u,\zeta)\widetilde N(du,d\zeta)\right|^r\right]^{1/r}\\
\le\ & E\left[\sup_{t\in[0,T]}\left|\int_0^t\sigma(u)dB(u)\right|^r\right]^{1/r}+E\left[\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}}\gamma(u,\zeta)\widetilde N(du,d\zeta)\right|^r\right]^{1/r}\\
\le\ & CE\left[\left(\int_0^T\sigma^2(u)du\right)^{r/2}\right]^{1/r}+CE\left[\left(\int_0^T\int_{\mathbb{R}}\gamma^2(u,\zeta)N(du,d\zeta)\right)^{r/2}\right]^{1/r}<\infty
\end{aligned}
$$

where C is a positive constant and the finiteness of the last expectations follows from the uniform boundedness of σ and γ. Due to (2.1) and to the boundedness of b, this suffices to prove the claim.

References

[1] K. Aase, B. Øksendal, N. Privault & J. Ubøe (2000): White noise generalizations of the Clark-Haussmann-Ocone theorem, with application to mathematical finance. *Finance and Stochastics* **4**, 465–496.

[2] F. Baghery & B. Øksendal (2007): A maximum principle for stochastic control with partial information. *Stochastic Analysis and Applications* **25**, 705-717.

[3] E. Barucci (2003): *Financial Markets Theory: Equilibrium, Efficiency and Information.* Springer, London.

[4] T. Björk, M. Davis & C. Landen (2010): Optimal investment under partial information. *Mathematical Methods of Operations Research* **71**, 371-399.

[5] F. Delbaen & W. Schachermayer (1994): A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* **300**, 463–520.

[6] F. Delbaen & W. Schachermayer (1995): Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probability Theory and Related Fields* **102**, 357–366.

[7] F. Delbaen & W. Schachermayer (1998): The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* **312**, 215–250.

[8] G. Di Nunno, B. Øksendal & F. Proske (2009): *Malliavin Calculus for Lévy Processes with Applications to Finance.* Springer , Berlin - Heidelberg.

[9] K.D. Elworthy, X.M. Li & M. Yor (1999): The importance of strictly local martingales; applications to radial Ornstein-Uhlenbeck processes. *Probability Theory and Related Fields* **115**, 325–355.

[10] H. Föllmer & P. Protter (2011): Local martingales and filtration shrinkage. *ESAIM: Probability and Statistics* **15**, S25–S38.

[11] C. Fontana & W.J. Runggaldier (2013): Diffusion-based models for financial markets without martingale measures. In: F. Biagini, A. Richter and H. Schlesinger (eds.), *Risk Measures and Attitudes*, 45–81, EAA Lecture Notes, Springer, London.

[12] J.M. Harrison & D. Kreps (1979): Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20**, 381–408.

[13] S. He, J. Wang & J. Yan (1992): *Semimartingale Theory and Stochastic Calculus.* CRC Press, Hong Kong.

[14] J. Jacod & A.N. Shiryaev (2003): *Limit Theorems for Stochastic Processes*, second edition. Springer, Berlin - Heidelberg.

[15] I. Karatzas & C. Kardaras (2007): The numèraire portfolio in semimartingale financial models. *Finance and Stochastics* **11**, 447–493.

[16] C. Kardaras (2012): Market viability via absence of arbitrage of the first kind. *Finance and Stochastics* **16**, 651–667.

[17] D. Kramkov & W. Schachermayer (1999): The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability* **9**(3), 904–950.

[18] D. Kreps (1981): Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics* **8**, 15–35.

[19] M. Loewenstein & G. Willard (2000): Local martingales, arbitrage and viability. *Economic Theory* **16**, 135–161.

[20] B. Øksendal & A. Sulem (2007): *Applied Stochastic Control of Jump Diffusions*, second edition. Springer, Berlin - Heidelberg - New York.

[21] B. Øksendal & A. Sulem (2012): Forward-backward stochastic differential games and stochastic control under model uncertainty. Online first in: *Journal of Optimization Theory and Applications*.

[22] A. Pascucci & W.J. Runggaldier (2012): *Financial Mathematics: Theory and Problems for Multi-period Models*. Springer, Milan.

[23] S. Pliska (1997): *Introduction to Mathematical Finance*. Blackwell, Oxford.

[24] M. Pratelli (1976): Sur certains espaces de martingales localement de carrè intègrable. *Sèminaire de Probabilitès* **X**, 401–413, Lecture Notes in Mathematics vol. 511, Springer.

[25] M.C. Quenez & A. Sulem (2012): BSDEs with jumps, optimization and applications to dynamic risk measures. To appear in: *Stochastic Processes and their Applications*.

[26] M. Rasony & L. Stettner (2005): On utility maximization in discrete-time financial market models. *Annals of Applied Probability* **15** (2), 1367–1395.

[27] D. Revuz & M. Yor (1999): *Continuous Martingales and Brownian Motion*. Springer, Berlin - Heidelberg - New York.

[28] W.J. Runggaldier (2003): Jump-diffusion models. In: S.T. Rachev (ed.), *Handbook of Heavy Tailed Distributions in Finance*, 169–209, Elsevier - North Holland, Amsterdam.

[29] W. Schachermayer (2001): Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability* **11**(3), 694–734.

[30] M. Schäl (2000): Price systems constructed by optimal dynamic portfolios. *Mathematical Methods of Operations Research* **51**, 375–397.

[31] M. Schäl (2000): Portfolio optimization and martingale measures. *Mathematical Finance* **10** (2), 289–303.

[32] H. Xing (2012): On backward stochastic differential equations and strict local martingales. *Stochastic Processes and their Applications* **122**, 2265–2291.

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